

11.2 Methodology For Determining Stress Intensity Factors

The linear elastic fracture mechanics approach to the analysis of cracked structures depends on the calculation of stress-intensity factors (K) for the typical crack geometries of interest.

The opening mode stress-intensity factor can always be expressed as

$$K_I = \beta\sigma\sqrt{\pi a} \quad (11.2.1)$$

where σ is the nominal stress remote from the crack and a is the crack size. The factor β is a function of crack geometry and of structural geometry. Since the dimension of K is $\text{ksi}\sqrt{\text{in.}}$ or equivalent, β must be dimensionless. For a central crack of length, $2a$, in an infinite sheet, the stress-intensity factor may be written

$$K_I = \sigma\sqrt{\pi a} \quad (11.2.2)$$

Comparison with Equation 11.2.1 shows that for an infinite sheet β is unity. Thus, β may be considered as a correction factor relating the actual stress-intensity factor to the central crack in an infinite sheet. The correction factors for various geometrical conditions under a given load condition may be combined in the form of a product to account for the increase or decrease in the stress-intensity factor.

As the linear elastic fracture mechanics approach to engineering problems became a typical design approach, a widespread need for stress-intensity factor solutions for typical geometries arose. This need was met by a series of handbooks which presented available solutions in a compact format. Some of these handbooks include

- Handbook of Stress Intensity Factor (Sih, 1973),
- The Stress Analysis of Cracks Handbook (Tada, et al., 1973),
- Compendium of Stress Intensity Factors (Rooke & Cartwright, 1976),
- Stress Intensity Factors Handbook (Murakami, 1987)

The handbook solutions, which are typically fundamental, may be extended to more complex cases through the principle of superposition or by compound analysis. The handbook solutions are also quite useful for bounding exact solutions as discussed in Section 11.4. When the structural geometry and loading system is fairly complicated, engineers normally resort to numerical analysis procedures (e.g., finite element analysis) which have been proven for their accuracy in establishing stress-intensity factors.

11.2.1 Principle of Superposition

Because the linear elastic fracture mechanics approach is based on elasticity, one can determine the effects of more than one type of loading on the crack tip stress field by linearly adding the stress-intensity factor due to each type of loading. The process of adding stress-intensity factor solutions for the same geometry is sometimes referred to as the principle of superposition. The only constraint on the summation process is that the stress-intensity factors must be associated with the same structural geometry, including crack geometry. Thus, stress-intensity factors associated with edge crack problems cannot be added to that of a crack growing radially from a

hole. An example will illustrate the conditions under which one might linearly add stress-intensity factors.

EXAMPLE 11.2.1 Axial and Bending Loads Combined

An edge crack of length a is subjected to a combination of axial and bending loads as shown. The stress-intensity factor for the edge crack geometry subjected to the tensile load (P) is given by

$$K_P = \frac{P}{BW} (\sqrt{\pi a}) \sec \beta \left(\frac{\tan \beta}{\beta} \right)^{\frac{1}{2}} \left[0.752 + 2.02 \left(\frac{a}{W} \right) + 0.37(1 - \sin \beta)^3 \right]$$

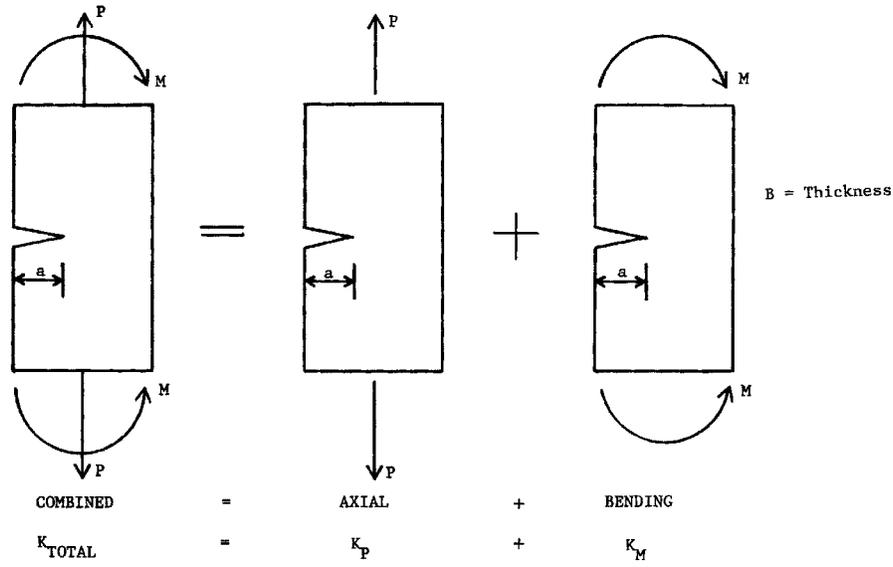
while that due to the bending moment (M) is given by

$$K_M = \frac{6M}{W^2 t} (\sqrt{\pi a}) \sec \beta \left(\frac{\tan \beta}{\beta} \right)^{\frac{1}{2}} \left[0.923 + 0.199(1 - \sin \beta)^4 \right]$$

with $\beta = \frac{\pi a}{2W}$

The stress intensity factor resulting from the combination of tensile and bending loads is given by the sum of K_P and K_M , so that

$$K_{TOTAL} = K_P + K_M$$



Edge Crack Geometry Loaded With Axial and Bending Loads

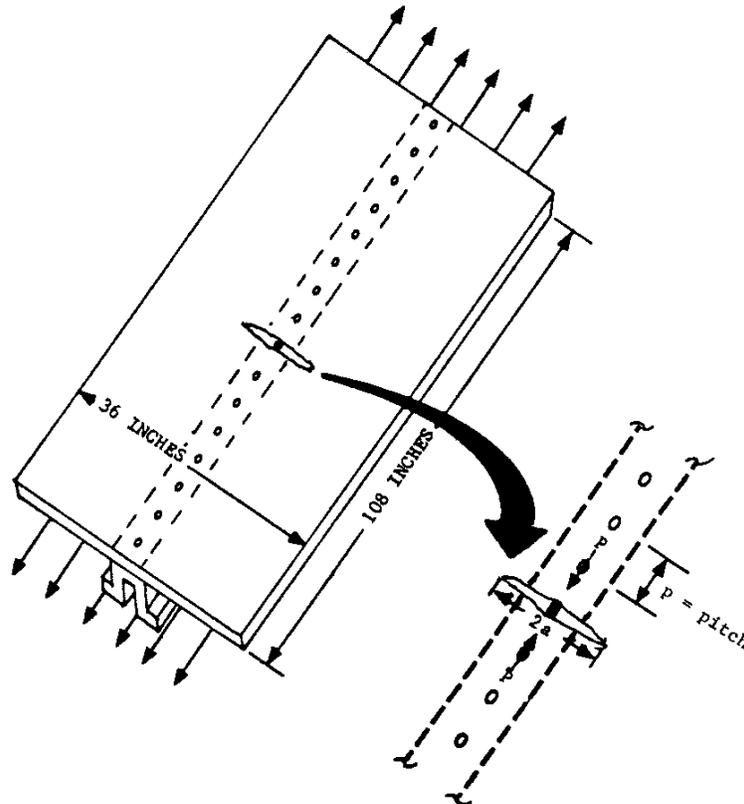
As shown by [Example 11.2.1](#), if the geometry of the structure is described, the effect of each loading condition can be separately determined and the effect of all the loading conditions can be obtained by summing the individual conditions, i.e.,

$$K_{TOTAL} = K_1 + K_2 + K_3 + \dots \quad (11.2.3)$$

This particular property is quite useful in the analysis of complex structures. [Example 11.2.2](#) (Wilhelm, 1970) further illustrates the principle of superposition.

EXAMPLE 11.2.2 Remote Loading and Concentrated Forces Combined

Many times in a particular aircraft design a part may develop cracks at rivet holes where the skin is attached to the frame or stringer. This situation is depicted in the figure below. It will be analyzed as a simple case in which the sheet is in uni-axial tension and the rivets above and below the crack are influential in keeping the crack closed. (Tests of panels with concentrated forces superimposed on the uniform tension loading simulate crack growth behavior in the presence of rivets.) The insert of the figure shows the local parameters necessary for determining the stress-intensity factors.

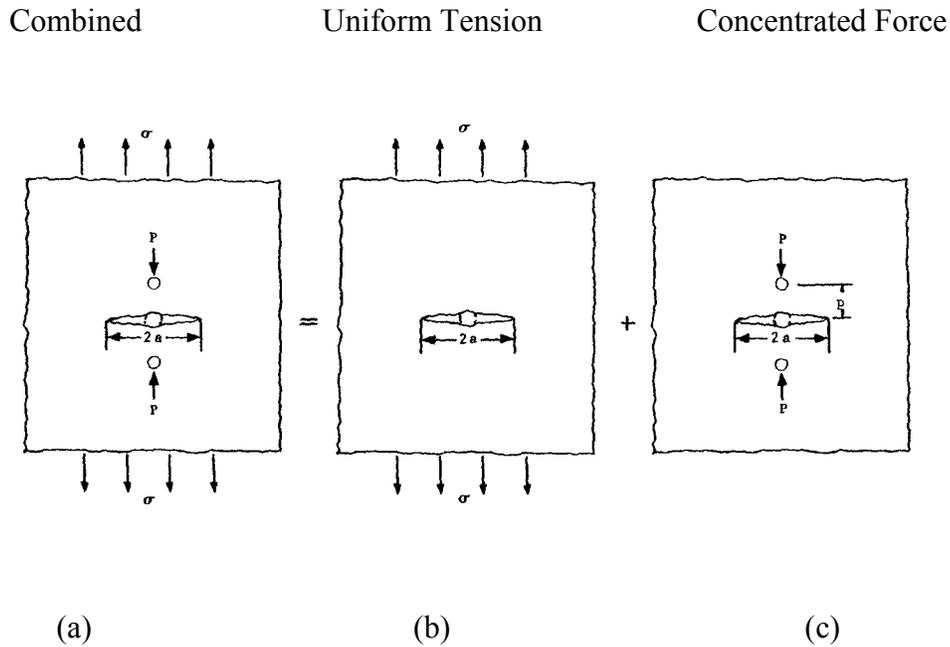


Crack at Rivet In a Riveted Skin-Stringer Panel (No Crack Buckling)

Assuming that a crack grows from the rivet hole, the total stress-intensity factor for this geometry is obtained using the linear superposition of stress-intensity factors. Closer examination of the figure indicates that the loading can be decomposed as shown in the next figure. The total stress-intensity factor is the sum of the remote loading and concentrated load induced stress-intensity factors.

Note: The concentrated force induced stress-intensity factor solution presented is only applicable if the concentrated forces are applied along the centerline of the sheet and at a distance greater than 3 or 4 times the hole diameter. Inasmuch as the concentrated forces are in an opposite direction to the uniform stress, and tend to close the crack, this stress-intensity is subtracted from the uniform extensional stress-intensity factor.

With knowledge of the stress-intensity solution for this geometry, it is possible to determine what effect the rivet closure forces have on the local stress field for similar problems.



$$K_{Total} = K(= \sigma \sqrt{\pi a}) + K_p \left(\frac{-P \sqrt{\pi a} (3 + \nu) p^2 + 2a^2}{2\pi B (a^2 + p^2)^{3/2}} \right)$$

Superposition of Stress intensities for Uniform Tension and Concentrated Force

In some cases, the additive property of the stress-intensity factor can be used to derive solutions for loading conditions that are not readily available. The process of deriving the stress-intensity factor for a center crack geometry, which is uniformly loaded with a pressure (p), shown in [Figure 11.2.1](#), illustrates this feature. [Figure 11.2.2](#) describes the process whereby the remotely loaded center crack geometry is decomposed into a set of two center crack geometries which have loading conditions, that when added, result in the canceling of the crack line loadings. The stress-intensity factor (K_I) for the plate loaded with the remote stress condition (σ) and the crack closing stresses (also equal to σ) is zero, i.e. $K_I = 0$, because the crack is clamped closed under such conditions. Thus, the equation for addition of stress-intensity factors

$$K_{TOTAL} = K_1 + K_2 \tag{11.2.4}$$

reduces to

$$K_{TOTAL} = 0 + K_2 \quad (11.2.5)$$

so that the stress-intensity factor for a pressurized center crack with pressure (p) equal to σ is the same as that associated with remote loading, i.e.

$$K_2 = \sigma\sqrt{\pi a} = K_{TOTAL} \quad (11.2.6)$$

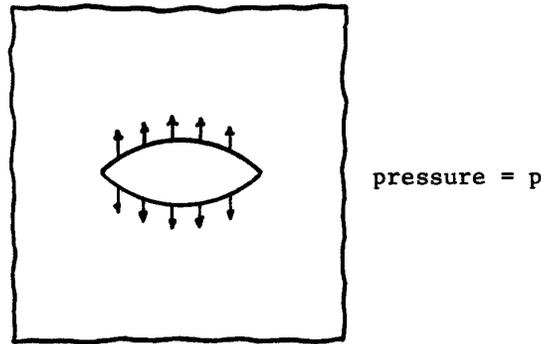


Figure 11.2.1. Internally Pressurized Center Crack

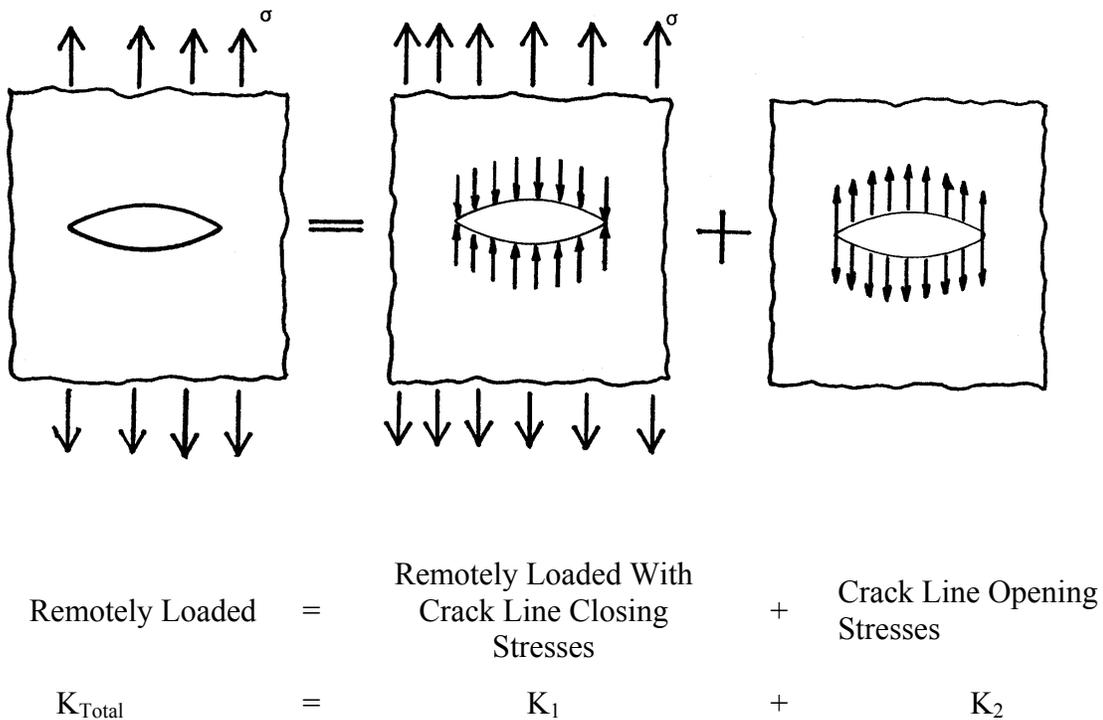


Figure 11.2.2. Principle of Superposition Illustrated for Center Cracked Geometry

Sometimes, it is difficult to visualize how one arrives at the values of the crack closing stresses. Consider the uncracked body with the uniformly applied remote loading as shown in [Figure 11.2.3a](#). Determination of the stresses along the dotted line lead to the observation that the stresses here are equal to the remote stress (σ). To obtain a stress-free condition along the dotted line, and thus simulate a cracked structural configuration, one must apply opposing stresses of

magnitude σ along the length of the dotted line as shown in [Figure 11.2.3b](#). The stresses along the dotted line generated by the applied remote stresses are the opening stresses ([Figure 11.2.3a](#)). The equal but opposite stresses are the crack closing stresses. The reader should note that the stresses on the dotted line that are generated by the remote loading lead to the crack opening condition; these opening stresses lead to non-zero values for the stress-intensity factor (see [Figure 11.2.2](#)).

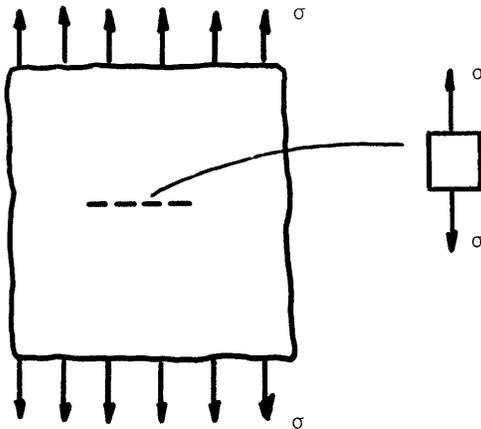


Figure 11.2.3a. Uniform Stresses Along Dotted Line Generated by Remote Loading

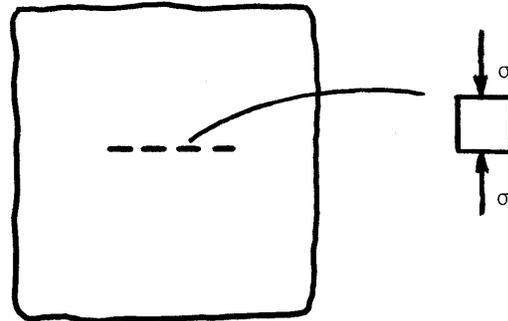


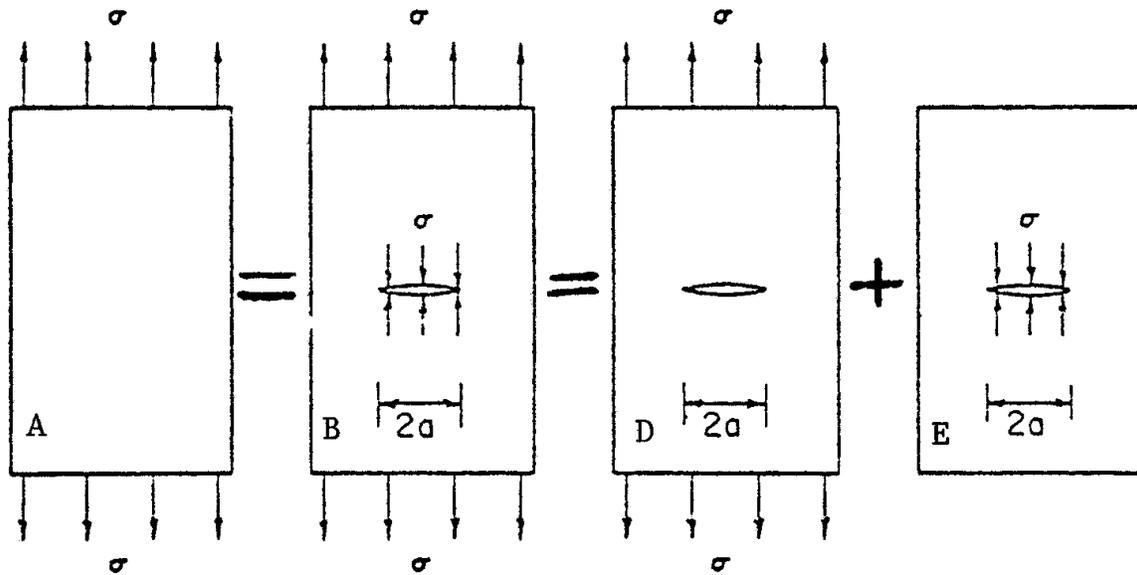
Figure 11.2.3b Opposing Stresses Applied Along the Dotted Line

[Figure 11.2.4](#) presents the concept of linear superposition of elastic solutions in a slightly different way so that the reader has a full appreciation of the procedure. The structural element B is noted to be exactly the same as element A; the crack closing stresses exactly balance the effect of the remote stresses along the line so the structural element B still experiences uniform tension throughout. Structural element B is further decomposed into elements D and E. Note that the crack loading stresses shown on the structural element E are crack closing stresses and, therefore, result in a stress-intensity factor which is the negative of the remotely applied loading case, i.e. $K_E = -K_D$.

Remote Loading W/O Crack

Remote Loading With Crack

Crack Loading Stresses On Crack



$$K_A = 0$$

$$K_B = 0$$

$$K_B = K_D + K_E = 0$$

$$K_E = -K_D$$

Figure 11.2.4. Illustration of Superposition Principle

Since K_D is known,

$$K_D = \sigma\sqrt{\pi a}$$

it follows that

$$K_E = -\sigma\sqrt{\pi a}$$

As we noted before, if the direction of stress in element E is reversed (becomes crack opening) then the stress-intensity factor is

$$K = \sigma\sqrt{\pi a}$$

The loading on structural element A in [Figure 11.2.5](#) can be decomposed into the series of loadings shown. The stress-intensity factor for element A is obtained from the superposition of the three other loadings:

$$K_A = K_B + K_D - K_E \tag{11.2.7}$$

Since it is obvious that the loadings in elements A and E will result in the same stress-intensity factor, i.e. $K_A = K_E$, the stress-intensity factor for element A becomes

$$K_A = 1/2[K_B + K_D] \tag{11.2.8}$$

The stress-intensity factors for elements B and D are known, i.e.

$$K_B = \sigma\sqrt{\pi a} \quad \text{and} \quad K_D = \left[\frac{W}{2a}\right]\sigma\sqrt{\pi a}$$

and, therefore

$$K_A = 1/2 \left[1 + \frac{W}{2a} \right] \sigma\sqrt{\pi a} \quad (11.2.9)$$

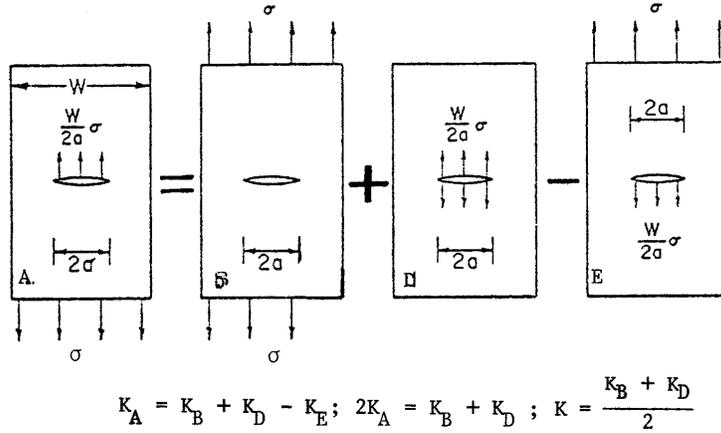


Figure 11.2.5. Application of Superposition Principle

Now a more complex example is presented using the principle of superposition applied in a two-step process. Shown in [Figure 11.2.6](#) is a structural element (F) in which intermediate values of load transfer occur through a pin loaded hole. As shown, Step 1 consists of decomposing element F into two parts, such that in one part the pin reacts its entire load and the other part is remotely loaded. The stress-intensity factor for element F is the sum of those generated by the decomposed elements, i.e.,

$$K_F = K_A + K_B^{\sigma_2}$$

where the superscript denotes the loading.

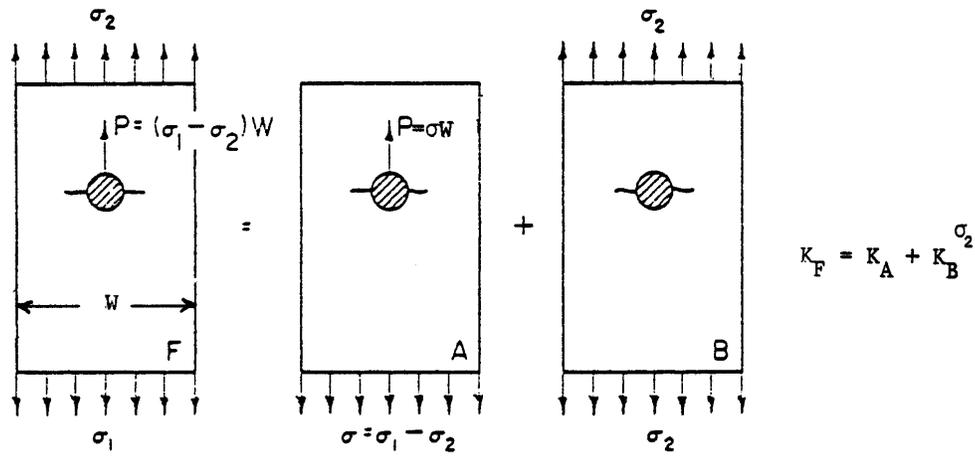
Step 2 involves the determination of K_A . The pin reactive loading on element A is decomposed into the loading shown in [Figure 11.2.6](#). Using the logic previously illustrated in [Figure 11.2.5](#), K_A is determined as

$$K_A = 0.5(K_B^{\sigma} + K_D^p)$$

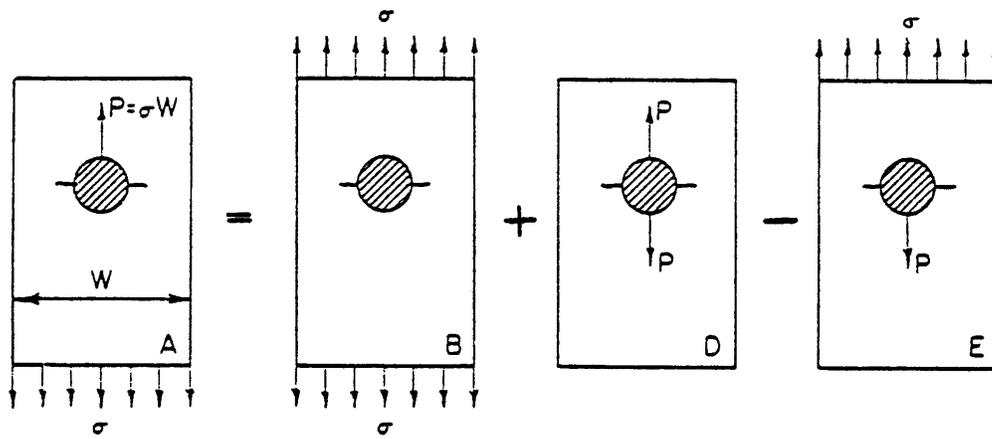
The stress-intensity factor for the loading on element F is

$$K_F = 0.5(K_B^{\sigma} + K_D^p) + K_B^{\sigma_2} \quad (11.2.10)$$

Note that while the stress-intensity factor solution formula for element B is the same in Steps 1 and 2, the stresses used in each calculation are different (as indicated by the superscripts).



Step 1: Decompose loading so that pin reacts its entire load.



Step 2: Decompose pin reactive loading. $K_A = \frac{K_B^{\sigma_2} + K_D^P}{2}$

Figure 11.2.6. Stress Intensity Factor for Pin-Loaded Hole (Bearing By-pass Problem) Obtained by Superposition

11.2.2 Developing Stress Intensity Factor Solutions

There are a number of methods that are available for developing stress-intensity factor solutions for crack body problems. Review articles and textbook chapters that summarize these methods are provided in Sneddon & Lowengrub (1969), Rice (1968), Paris & Sih (1965), Sih (1973a, 1973b), Tada, et al. (1973), Rooke & Cartwright (1976), Wilhelm (1970), Parker (1981), Broek (1974) and Goodier (1969). The basic solutions for simple geometries can be derived by means of classical methods of elasticity which employ complex stress functions Sneddon & Lowengrub (1969), Rice (1968), Westergaard (1939) and Mushkelishvili (1953).

For finite size bodies containing cracks, the boundary conditions usually prohibit a closed form solution. In such cases, numerical solutions can be obtained using methods such as the finite element method, the boundary collocation technique [Gross, et al., 1964; Newman, 1971], or the boundary integral method [Cruse, 1972; Cruse & Besuner, 1975]. Solutions for multiple load path geometries can sometimes be obtained from basic stress field solutions combined with displacement compatibility requirements for all the structural members involved [Swift & Wang, 1969]. Section 4 describes this method and provides an example based on the displacement compatibility method.

There are also several experimental methods that have been used to obtain (or verify) the stress-intensity factor for cracked structural members. These experimental methods include: The compliance method, the photoelastic method [Smith, 1975; Kobayashi, 1973], the fatigue crack growth (inverse) method [James & Anderson, 1969; Grandt & Hinnericks, 1974; Gallagher, et al., 1974], and the interferometric method [Packman, 1975; Pitoniak, et al., 1974].

While a general knowledge of each stress-intensity factor solution method might be useful for attacking specific problems, detailed knowledge is required before any method can be applied to solve a given problem. Beyond what is described elsewhere in these guidelines, an engineer can also utilize two separate solution techniques to solve any two-dimensional structural geometry or loading situation without access to a damage tolerant specialist. One solution technique involves the generation of the stress for an uncracked body along the expected path of crack propagation. (The finite element method provides a powerful tool for generating stress at any point in an uncracked body). The second solution technique involves the generation of the stress-intensity factor solution via an integral calculation that employs the stresses obtained for the case of the uncracked body along the expected path of the crack. Two integral calculation technique types are available: the Green's function technique [Cartwright & Rooke, 1979, 1978; Cartwright, 1979; Hsu & Rudd, 1978; Hsu, et al., 1978] and the weight function technique [Cartwright & Rooke, 1978; Cartwright, 1979; Bueckner, 1971; Rice, 1972; Grandt, 1975]. These two crack-line loading techniques are reviewed in the following subsections.

11.2.2.1 Green's Function Technique

The Green's function technique takes advantage of the additive property of the stress-intensity factor and is based on generalized point load solutions of crack problems. For example, the point load solution for the central crack problem described in [Figure 11.2.7](#) is given by:

$$K = \frac{P}{B} \frac{1}{\sqrt{\pi a}} \sqrt{\left(\frac{a+x}{a-x}\right)} \quad (11.2.11)$$

This solution can be used to obtain the stress-intensity factor for stresses distributed over the crack faces by noting that the point load per unit thickness (P/B) in Equation 11.2.11 can be replaced by the product of the pressure stress ($\sigma(x)$) and the distance over which it acts (dx). Thus, the stress-intensity factor for the distributed stresses applied to the crack becomes:

$$K = \int_{-a}^{+a} \frac{\sigma dx}{\sqrt{\pi a}} \frac{a+x}{\sqrt{a^2-x^2}} \quad (11.2.12)$$

The stress-intensity factor for the case of uniform opening stresses applied to the crack, where σ is a constant, is determined to be $K = \sigma\sqrt{\pi a}$, as was expected from the discussion of the method of superposition described previously.

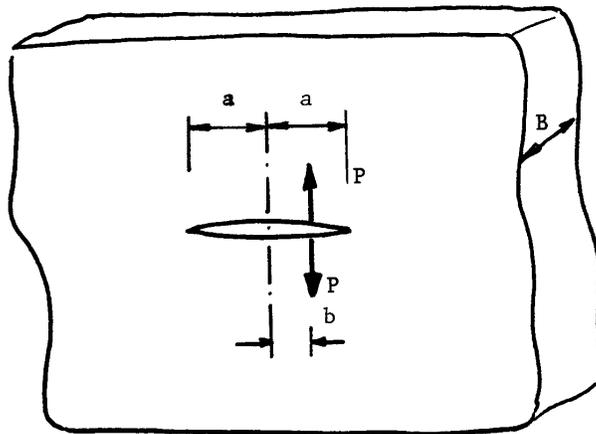


Figure 11.2.7. Point Load (P) Applied to the Crack Faces for a Central Crack Located in an Infinite Plate

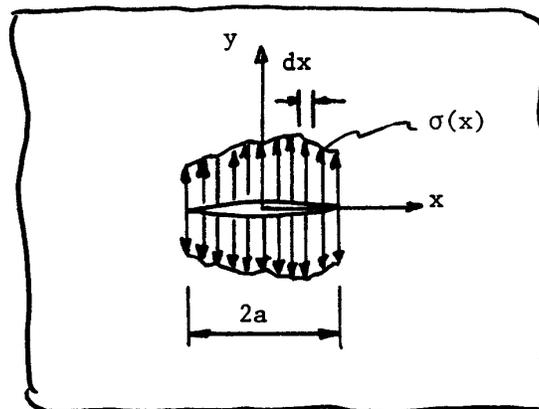


Figure 11.2.8. Distributed Loading Applied to Crack Faces of the Central Crack

Given a point force solution for a geometry of concern, it is then possible to define the summation process that would integrate the effects of stress loading over the crack faces. Integral equations such as that defined by Equation 11.2.12 utilize the stress solutions from the uncracked body problem. A number of point force stress-intensity factor solutions are presented in the tables given in Section 11.3 and an extensive review of the availability and application of Green's functions can be found in Cartwright & Rooke [1979]. Other reviews can be found in Cartwright & Rooke [1978] and Cartwright [1979].

One of the cases reviewed by Cartwright and Rooke [Cartwright & Rooke, 1979] is of particular interest to structural engineers. They presented the work by Hsu and Rudd [1978] on the development of a Green's function for a diametrically cracked hole. The Hsu and Rudd Green's function was based on a series of finite-element determined stress-intensity factor solutions for a symmetrical set of point forces of the type shown in [Figure 11.2.9](#). The finite-element point force solutions were developed as a function of position for $X (=x/a) < 0.9$ and a limiting expression was given for $X > 0.9$. The Hsu and Rudd Green's function is shown in [Figure 11.2.10](#) for several values of a/R ; also shown are Green's functions for an edge crack and for a central crack. Note that all the Green's functions tend to infinity as X approaches 1. It should also be noted that the Green's functions presented are based on the following format

$$K = \frac{1}{\sqrt{\pi a}} \int_0^a \sigma(x) G(x, a) dx \quad (11.2.13)$$

which has been widely used. Hsu and Rudd based their presentation of the Green's function on an approach taken by Hsu, et al. [1978], wherein the Green's function $G(x, a)$ in Equation 11.2.13 is obtained by multiplying the Hsu, et al. value G^H by π , i.e.

$$G(x, a) = \pi G^H(x, a) \quad (11.2.14)$$

The complete table of $G^H(x, a)$ derived by Hsu, et al. can be found in [Table 11.2.1](#). Other work by Hsu and co-workers on lug-type problems can be found in Section 11.3.

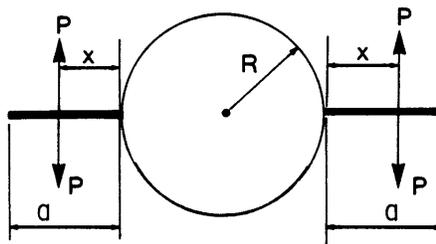


Figure 11.2.9. Diametrically Cracked Hole With Symmetrically Located Point Focus

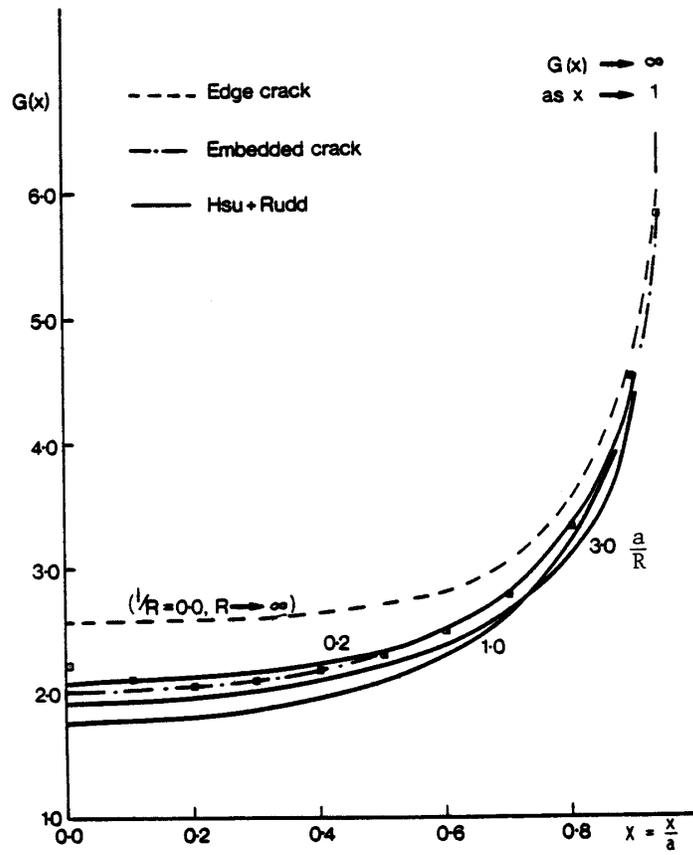


Figure 11.2.10. Green's Function for Geometry and Loading Described in Figure 11.2.9 [Cartwright & Rooke, 1979; Hsu & Rudd, 1978; Hsu, et al., 1978]

Table 11.2.1. Green's Function For A Double Crack Emanating From An Open Hole In An Infinite Plate [Hsu, et al., 1978]

x/a	a/r													
	.20	.30	.40	.50	.60	.70	.80	.90	1.00	1.40	1.60	2.00	2.40	3.00
.00	.664	.629	.603	.595	.568	.575	.572	.554	.548	.571	.582	.594	.600	.611
.10	.676	.639	.615	.604	.582	.583	.586	.569	.563	.587	.596	.603	.603	.615
.20	.688	.645	.628	.617	.599	.596	.600	.589	.578	.604	.612	.613	.609	.624
.30	.699	.658	.646	.633	.621	.613	.623	.610	.598	.627	.630	.625	.619	.639
.40	.718	.679	.671	.656	.651	.639	.655	.639	.624	.656	.653	.642	.635	.664
.45	.740	.691	.689	.671	.673	.657	.674	.658	.643	.674	.665	.654	.647	.680
.50	.760	.708	.712	.689	.698	.681	.701	.682	.668	.692	.678	.670	.662	.699
.55	.781	.732	.739	.712	.730	.711	.733	.708	.699	.709	.695	.692	.679	.723
.60	.802	.764	.762	.746	.770	.752	.766	.739	.737	.730	.725	.721	.702	.753
.70	.889	.868	.837	.838	.865	.867	.850	.827	.847	.819	.801	.811	.760	.842
.75	.960	.946	.907	.911	.913	.960	.912	.911	.929	.888	.859	.884	.817	.905
.80	1.071	1.089	1.044	1.030	.989	1.056	1.018	.995	1.021	.985	.955	.979	.904	.977
.85	1.234	1.254	1.245	1.211	1.141	1.252	1.177	1.187	1.192	1.130	1.130	1.120	1.042	1.101
.90	1.429	1.432	1.434	1.436	1.437	1.438	1.440	1.441	1.442	1.445	1.446	1.448	1.449	1.451

$$*For x/a > 0.9, G^H(a/r, x/a) = \frac{1}{\pi} \left[\frac{4 \left(1 + \frac{a}{r}\right)}{\left(1 - \frac{x}{a}\right) \left(1 + \frac{x}{a} + \frac{2r}{a}\right)} \right]^{\frac{1}{2}}$$

There are two cautionary remarks that must be made about the use of Green's function techniques for solving crack problems. First, if all the loading across the crack tip is not tensile, and if the stress-intensity factor is positive at the crack tip of interest, the crack faces at some distance away from the crack tip may have (mathematically) merged in a nonphysical overlapping manner and the estimated stress-intensity factor might be unconservatively low. Accordingly, one should check to determine if the crack displacements all along the crack are positive and thus non-overlapping to ensure validity of the solution. Second, it is important in displacement boundary value problems to derive a Green's function that accounts for the requirement that there be zero displacement on those boundaries where displacement conditions are applied when estimating the stress-intensity factor from the uncracked geometry solution. Typically, neglecting this requirement for displacement boundary value problems produces a stress-intensity factor that is conservatively high. These two cautions apply equally well to the weight function technique.

11.2.2.2 The Weight Function Technique

The weight function technique can be derived using the definition of the strain energy release rate [Parker, 1981; Cartwright, 1979; Bueckner, 1971; Rice, 1972]. The stress-intensity factor is

obtained from the difference between the strain energy of a cracked structure and of the identical structure without a crack, and is given by:

$$K_I = \int_a \sigma(x)m(x,a)dx \quad (11.2.15)$$

where the function $m(x,a)$ is the Bueckner weight function, a function which is unique for the given geometry and is independent of the loading from which it was derived. The weight function is defined as a function of

- 1) material properties,
- 2) a known stress-intensity factor (K^*) for the given geometry under a defined loading, and
- 3) the crack opening $v^*(x,a)$ corresponding to K^* :

$$m(x,a) = \frac{H}{2K^*} \frac{\partial v^*}{\partial a}(x,a) \quad (11.2.16)$$

H is a material constant that is given by:

$$H = \frac{8\mu}{1+\kappa} \quad \begin{array}{l} =E \text{ for plane stress} \\ =\frac{E}{1-\nu^2} \text{ for plane strain} \end{array} \quad (11.2.17)$$

with μ = shear modulus and κ is defined as a function of the stress state and Poisson's ratio (ν)

$$\kappa = \begin{array}{l} \frac{3-\nu}{1+\nu} \text{ for plane stress} \\ 3-4\nu \text{ for plane strain} \end{array} \quad (11.2.18)$$

For the infinite plate center crack problem K^* , v^* , $\frac{\partial v^*}{\partial a}$, and m are given by the following equations:

$$K^* = \sigma\sqrt{\pi a} \quad (11.2.19)$$

$$v^*(x,a) = \left(\frac{1+\kappa}{4\mu}\right) \sigma\sqrt{a^2-x^2} \quad \text{for } -a \leq x \leq a \quad (11.2.20)$$

$$\frac{\partial v^*}{\partial a} = \left(\frac{1+\kappa}{4\mu}\right) \sigma a \frac{1}{\sqrt{a^2-x^2}} \quad (11.2.21)$$

and

$$m(x,a) = \sqrt{\frac{a}{\pi}} \frac{1}{\sqrt{a^2-x^2}} \quad (11.2.22)$$

The stress-intensity factor associated with a symmetrical pressure loading of $\sigma(x)$ on the central crack faces is then given by

$$K = \sqrt{\frac{a}{\pi}} \int_{-a}^{+a} \frac{\sigma(x)}{\sqrt{a^2 - x^2}} dx \quad (11.2.23)$$

The reader is cautioned to note that Equations 11.2.23 and 11.2.12 differ. However, both equations yield exactly the same stress-intensity factor solution when the pressure stress σ is a symmetrical function, i.e., the stress at $x = x_o$ is equal to the stress at $x = -x_o$ ($0 \leq x_o \leq a$). The reason that Equations 11.2.23 and 11.2.12 differ is that the Bueckner function in Equation 11.2.12 was derived for a symmetrical loading whereas the Green's function was derived for the more general case of unsymmetrical loading. Thus, when deriving the weight function one should seek to locate stress-intensity factor (K^*) and crack displacement (v^*) solutions which are representative of the loading symmetry associated with the problems that are to be solved.

A weight function for radially and diametrically cracked holes was developed by Grandt [1975] for through-thickness type cracks. His solution is given by

$$K = \frac{H}{K_B} \int_0^a \sigma(x) \frac{\partial \eta}{\partial a} dx \quad (11.2.24)$$

where K_B represents the appropriate (radial or diametrical) Bowie stress-intensity factor (Section 11.3), and the crack opening displacement η was obtained from finite-element solutions. The displacements η were described by the conic section equation:

$$\left(\frac{\eta}{\eta_o} \right)^2 = \frac{2}{2+m} \left(\frac{a-x}{a} \right) + \frac{m}{2+m} \left(\frac{a-x}{a} \right)^2 \quad (11.2.25)$$

Here η_o is the displacement at the crack mouth ($x=0$) and m is the conic section coefficient from

$$m = \pi \left[\frac{H\eta_o}{2\sigma a Y} \right]^2 - 2 \quad (11.2.26)$$

In this instance, Y is the Bowie geometric factor

$$Y = \frac{K_B}{\sigma \sqrt{a}} \quad (11.2.27)$$

The finite-element results for the crack mouth displacement η_o were closely represented by the least squares expression

$$\eta_o = R \sum_{i=0}^6 D_i (a/R)^i \quad (11.2.28)$$

where the coefficients D_i are given in [Table 11.2.2](#).

Table 11.2.2. Least Squares Fit Of Finite Element Data For Crack Mouth Displacement [Grandt, 1975]

$$\eta_0 = R \sum_{i=0}^6 D_i (a/R)^i$$

Coefficient	Single Crack	Double Crack
D ₀	-1.567 x10 ⁻⁶	1.548 x10 ⁻⁵
D ₁	6.269 x10 ⁻⁴	5.888 x10 ⁻⁴
D ₂	-6.500 x10 ⁻⁴	-4.497 x10 ⁻⁴
D ₃	4.466 x10 ⁻⁴	3.101 x10 ⁻⁴
D ₄	-1.725 x10 ⁻⁴	-1.162 x10 ⁻⁴
D ₅	3.485 x10 ⁻⁵	2.228 x10 ⁻⁵
D ₆	-2.900 x10 ⁻⁶	-1.694 x10 ⁻⁶

Grandt has applied the weight function technique to a number of fastener-type cracked hole problems. Using finite-element descriptions of the stress along the expected crack path for a hole that has been cold-worked (loaded) to a 0.006 inch radial expansion and then unloaded, Grandt was able to derive the stress-intensity factor shown in [Figure 11.2.11](#) for a remote stress loading of 40 ksi. [Figure 11.2.11](#) also provides the stress-intensity factor solution for a remote stress loading of 40 ksi applied to a radially cracked hole without cold-working. The dramatic difference in stress-intensity factors from the two cases has been shown to translate itself into orders of magnitude difference in crack growth rate behavior.

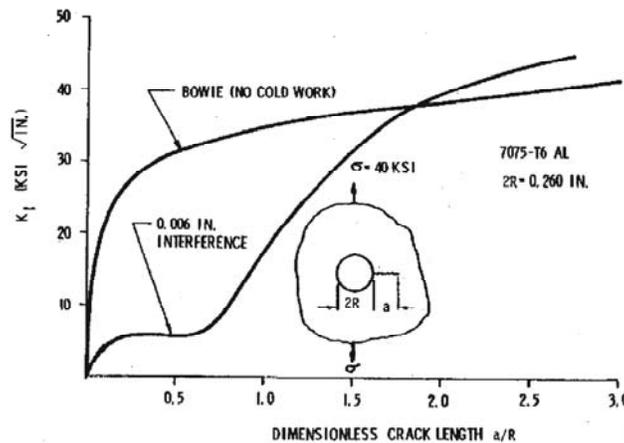


Figure 11.2.11. Stress-Intensity Factor Calibration for a Cold Worked Hole [Grandt, 1975]

11.2.3 Finite Element Methods

In all cases where an expression for the stress-intensity factor cannot be obtained from existing solutions, finite-element analysis can be used to determine K [Chan, et al., 1970; Byskov, 1970; Tracey, 1971; Walsh, 1971]. Certain aircraft structural configurations have to be analyzed by finite-element techniques because of the influence of complex geometrical boundary conditions or complex load transfer situations. In the case of load transfer, the magnitude and distribution

of loadings may be unknown. With the application of finite-element methods, the required boundary conditions and applied loadings must be imposed on the model.

Complex structural configurations and multicomponent structures present special problems for finite-element modeling. These problems are associated with the structural complexity. When they can be solved, the stress-intensity factor is determined in the same way as in the case of simpler geometry. This subsection deals with the principles and procedures that permit the determination of the stress-intensity factor from a finite-element solution.

Usually quadrilateral, triangular, or rectangular constant-strain elements are used, depending on the particular finite-element structural analysis computer program being used. For problems involving holes or other stress concentrations, a fine-grid network is required to accurately model the hole boundary and properly define the stress and strain gradients around the hole or stress concentration.

Within the finite-element grid system of the structural problem, the crack surface and length must be simulated. Usually, the location and direction of crack propagation is perpendicular to the maximum principal stress direction. If the maximum principal stress direction is unknown, then an uncracked stress analysis of the finite-element model should be conducted to establish the location of the crack and the direction of propagation.

The crack surfaces and lengths are often simulated by double-node coupling of elements along the crack line. Progressive crack extension is then simulated by progressively “unzipping” the coupled nodes along the crack line. Because standard finite-element formulations do not treat singular stress behavior in the vicinity of the ends of cracks, special procedures must be utilized to determine the stress-intensity factor. Three basic approaches to obtain stress-intensity factors from finite-element solutions have been rather extensively studied. These approaches are as follows:

- a) Direct Method. The numerical results of stress, displacement, or crack-opening displacement are fitted to analytical forms of crack-tip-stress-displacement fields to obtain stress-intensity factors.
- b) Indirect Method. The stress-intensity follows from its relation to other quantities such as compliance, elastic energy, or work energy for crack closure.
- c) Cracked Element. A hybrid-cracked element allowing a stress singularity is incorporated in the finite-element grid system and stress-intensity factors are determined from nodal point displacements along the periphery of the cracked element.

These approaches can be applied to determine both Mode 1 and Mode 2 stress-intensity factors. Application of methods has been limited to two-dimensional planar problems. The state-of-the-art for treating three-dimensional structural crack problems is still a research area.

11.2.3.1 Direct Methods

The direct methods use the results of the general elastic solutions to the crack-tip stress and displacement fields. For the Mode 1, the crack tip stresses can always be described by the equations

$$\begin{aligned}
\sigma_x &= \frac{K_1}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left[1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right] \\
\sigma_y &= \frac{K_1}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left[1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right] \\
\sigma_z &= 0 \quad \text{for plane stress} \\
\sigma_z &= \nu(\sigma_x + \sigma_y) \quad \text{plane strain} \\
\tau_{xy} &= \frac{K_1}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \\
\tau_{xz} &= \tau_{yz} = 0
\end{aligned} \tag{11.2.29}$$

where r and θ are polar coordinates originating at the crack tip, and where x is the direction of the crack, y is perpendicular to the crack in the plane of the plate, and z is perpendicular to the plate surface.

If the stresses around the crack tip are calculated by means of finite-element analysis, the stress-intensity factor can be determined as

$$K_1 = \sigma_{ij} \frac{\sqrt{2\pi r}}{f_{ij}(\theta)} \tag{11.2.30}$$

where i and j are used to represent various permutations of x and y .

By taking the stress calculated for an element not too far from the crack tip, the stress intensity follows from a substitution of this stress and the r and θ of the element into Equation 11.2.30. This can be done for any element in the crack tip vicinity.

Ideally, the same value of K should result from each substitution; however, the stress field equations are only valid in an area very close to the crack tip. Also at some distance from the crack tip, nonsingular terms should be taken into account. Consequently, the calculated K differs from the actual K . The result can be improved [Chan, et al., 1970] by refining the finite-element mesh or by plotting the calculated K as a function of the distance of the element to the crack tip. The resulting line should be extrapolated to the crack tip, since the crack tip equations are exact for $r = 0$. Usually, the element at the crack tip should be discarded. Since it is too close to the singularity, the calculated stresses are largely in error. As a result, Equation 11.2.30 yields a K value that is more in error than those for more remote element, despite the neglect of the nonsingular terms.

Instead of the stresses, one can also use the displacements for the determination of K . In general, the displacements of the crack edge (crack-opening displacements) are employed. The Mode 1 and Mode 2 plane strain displacement equations are given by

$$u_1 = \frac{2K_1(1+\nu)}{E} \left[\frac{r}{2\pi} \right]^{1/2} \cos \frac{\theta}{2} \left(1 - 2\nu + \sin^2 \frac{\theta}{2} \right) \tag{11.2.31}$$

$$v_1 = \frac{2K_1(1+\nu)}{E} \left[\frac{r}{2\pi} \right]^{1/2} \sin \frac{\theta}{2} \left(2 - 2\nu - \cos^2 \frac{\theta}{2} \right)$$

and by

$$u_2 = \frac{2K_2(1+\nu)}{E} \left[\frac{r}{2\pi} \right]^{1/2} \sin \frac{\theta}{2} \left(2 - 2\nu + \cos^2 \frac{\theta}{2} \right)$$

$$v_2 = \frac{2K_2(1+\nu)}{E} \left[\frac{r}{2\pi} \right]^{1/2} \cos \frac{\theta}{2} \left(2\nu - 1 + \sin^2 \frac{\theta}{2} \right)$$
(11.2.32)

respectively. The functions u and v represent the displacements in the x and y direction, respectively. The crack tip polar coordinates r and θ are chosen to coincide with the nodal points in the finite element mesh where displacements are desired. Since the above elastic field equations are only valid in an area near the tip of the crack, the application should be restricted to that area.

11.2.3.2 Indirect Methods

The indirect methods use relationships that exist between the stress-intensity factor (K) and the elastic-energy content (U) of the cracked structure. These relationships are developed in Section 1.3.2 along with a full discussion of the strain energy release rate (G) and compliance (C), i.e. the inverse stiffness of the system. The stress-intensity factor is related to these parameters by the following:

$$K^2 = G\bar{E} \tag{11.2.33}$$

$$K^2 = \frac{\partial U}{\partial a} \cdot \frac{\bar{E}}{B} \tag{11.2.34}$$

$$K^2 = \frac{P^2}{2B} \cdot \frac{\partial C}{\partial a} \cdot \bar{E} \tag{11.2.35}$$

where B is the plate thickness and \bar{E} is the elastic modulus E in plane stress and is $E/(1-\nu^2)$ in plane strain.

The elastic energy content and the compliance of cracked structures are obtained for a range of crack sizes either by solving the problem for different crack sizes or by unzipping nodes. Differentiation with respect to crack size gives K from the above equations. The advantage of the elastic-energy content and compliance methods is that a fine mesh is not necessary, since accuracy of crack-tip stresses is not required. A disadvantage is that differentiation procedures can introduce errors.

The strain energy release rate relationship (Equation 11.2.33) was derived based on the use of the crack tip stress field and displacement equations to calculate the work done by the forces required to close the crack tip. The crack tip closing work can be calculated by uncoupling the next nodal point in front of the crack tip and by calculating the work done by the nodal forces to close the crack to its original size.

The concept is that if a crack were to extend by a small amount, Δa , the energy absorbed in the process is equal to the work required to close the crack to its original length. The general integral equations for strain energy release rates for Modes 1 and 2 deformations are

$$G_1 = \Delta a \xrightarrow{\text{lim} \rightarrow 0} \frac{1}{2\Delta a} \int_0^{\Delta a} \sigma_y(\Delta a - r, 0)v(r, \pi)dr, \quad (11.2.36)$$

$$G_2 = \Delta a \xrightarrow{\text{lim} \rightarrow 0} \frac{1}{2\Delta a} \int_0^{\Delta a} \tau_{xy}(\Delta a - r, 0)u(r, \pi)dr.$$

The significance of this approach is that it permits an evaluation of both K_I and K_{II} from the results of a single analysis.

In finite-element analysis, the displacements have a linear variation over the elements and the stiffness matrix is written in terms of forces and displacements at the element corners or nodes. Therefore, to be consistent with finite-element representation, the approach for evaluating G_I and G_{II} is based on the nodal-point forces and displacements. An explanation of application of this work-energy method is given with reference to [Figure 11.2.12](#). The crack and surrounding elements are a small segment from a much larger finite-element model of a structure. In terms of the finite-element representation, the amount of work required to close the crack, Δa , is one-half the product of the forces at nodes c and d and the distance ($v_c - v_d$) which are required to close these nodes. The expressions for strain energy release rates in terms of nodal-point displacements and forces are (see [Figure 11.2.12](#) for notations)

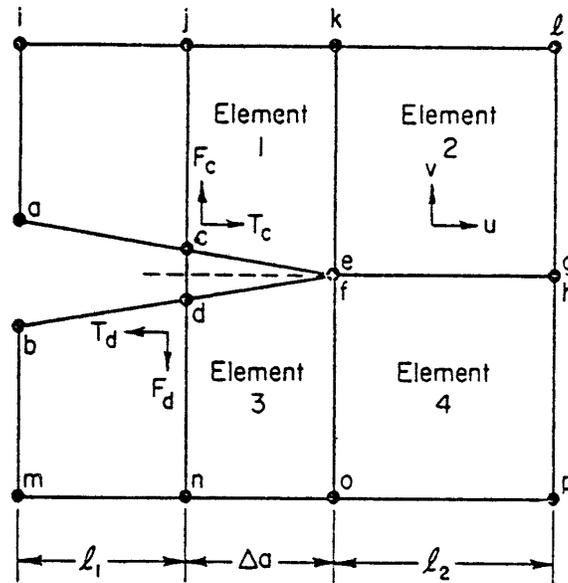


Figure 11.2.12. Finite-Element nodes Near Crack Tip.

$$G_1 = \Delta a \xrightarrow{\text{lim} \rightarrow 0} \frac{1}{2\Delta a} \overline{F_c}(v_c - v_d) \quad (11.2.37)$$

$$G_2 = \Delta a \xrightarrow{\text{lim} \rightarrow 0} \frac{1}{2\Delta a} \bar{T}_c(u_c - u_d)$$

11.2.3.3 Cracked Element Methods

This approach involves the use of a hybrid-cracked element that is incorporated into a finite-element structural analysis program. To date, only two dimensional crack problems can be solved with the cracked-element approach. Elements have been developed [Byskov, 1970; Tracey, 1971; Walsh, 1971; Gallagher, 1978; Jordon, et al., 1973; Atluri, et al., 1974; Hellen, 1979] that allow a stress singularity to occur at the crack tip.

The cracked element consists of boundary nodal points around the geometrical boundary of the element. The element is either contained within the complete finite-element model or is solved separately using the results of finite-element analysis. In either case, the crack surface is simulated by unzipping a double-noded line along the line of expected crack extension. This builds into the structural model the proper stiffness due to the presence of the crack. The variation of stress-intensity factors (K_1 and K_2) with crack length is determined by progressively unzipping the sets of coupled nodes.

Studies have been conducted on the variation of stress-intensity factors with cracked-element size and location [Jordon, et al., 1973; Atluri, et al., 1974]. These results define some definite guidelines in using cracked-element models. First, the distance from the crack tip to the cracked-element nodal points should be as constant as possible. Secondly, for long edge-cracks or cracks emanating from holes, the cracked element should only contain an area very near the crack tip.